J. Math. Anal. Appl. 379 (2011) 171–179

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Differential and integral equations with Henstock–Kurzweil integrable functions

S. Heikkilä

Department of Mathematical Sciences, University of Oulu, Box 3000, FIN-90014 University of Oulu, Finland

ARTICLE INFO

Article history:
Received 12 September 2010
Available online 24 December 2010
Submitted by Steven G. Krantz

Keywords:
Henstock–Kurzweil integral
Banach space
Banach lattice
Weakly complete
Order cone
Normal
Fully regular
Fixed point
Integral equation
Cauchy problem
Solution

ABSTRACT

In this paper we apply fixed point theorems for increasing mappings in ordered normed spaces to prove existence and comparison results for solutions of discontinuous functional differential and integral equations containing Henstock–Kurzweil integrable functions.

1. Introduction

The Henstock–Kurzweil integral provides a tool for integration of highly oscillatory functions which occur in quantum theory and nonlinear analysis. It is also easy to understand because its definition requires no measure theory. Moreover, all Bochner integrable (in real-valued case Lebesgue integrable) functions are Henstock–Kurzweil (shortly HK) integrable, but not conversely. For instance, HK integrability encloses improper integrals. The real-valued function f defined on $[0,1]$ by $f(0) = 0$ and $f(t) = t^2\cos(1/t^2)$ is differentiable on $[0,1]$, and f' is HK integrable. But f' is not Lebesgue integrable on $[0,1]$. More generally, let t be called a singular point of the domain interval of a real-valued function being not Lebesgue integrable on any interval that contains t. Then (cf. [12]) there exist HK "integrable functions on an interval that admit a set of singular points with its measure as close as possible but not equal to that of the whole interval."

In this paper we apply recently proved fixed point theorems in ordered normed spaces to solve equations that contain HK integrable Banach lattice-valued functions and discontinuous nonlinearities.

The paper is organized as follows. In Section 2 we present basic concepts of ordered normed spaces, and a preliminary theory of HK integrable functions from a compact real interval into a Banach space. In Section 3 a fixed point theorem in ordered spaces is applied to prove existence and comparison results for solutions of a Volterra integral equation in weakly sequentially complete Banach lattices. The integrals in the considered equations are Henstock–Kurzweil integrals. The functions in these equations are allowed to be discontinuous and depend functionally on the unknown function. Applications to Cauchy problems are given in Section 4. The obtained results are illustrated by an example.

E-mail address: sheikki@cc.oulu.fi.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.12.050

2. Preliminaries

We shall first present some basic properties of ordered normed spaces.

A closed subset X_+ of a normed space X is called an *order cone* if $X_+ + X_+ \subseteq X_+$, $X_+ \cap (-X_+) = \{0\}$ and $cX_+ \subseteq X_+$ for each $c \geqslant 0$. It is easy to see that the order relation $\leqslant$, defined by

$$x \leqslant y \quad \text{if and only if} \quad y - x \in X_+,$$

is a partial ordering in X, and that $X_+ = \{y \in X \mid 0 \leqslant y\}$. The space X, equipped with this partial ordering, is called an *ordered normed space*. The order interval $[y, z] = \{x \in X \mid y \leqslant x \leqslant z\}$ is a closed subset of X. We say that an order cone X_+ of a normed space X is *normal* if there is such a constant $\gamma \geqslant 1$ that

$$0 \leqslant x \leqslant y \quad \text{in } X \quad \text{implies} \quad \|x\| \leqslant \gamma \|y\|. \tag{2.1}$$

X_+ is called *fully regular* if all increasing and (norm) bounded sequences of X_+ converge. As for the proof of the following result, see, e.g., [5, Theorems 2.2.1 and 2.4.5].

Lemma 2.1. *Let X_+ be an order cone of a Banach space X. If X_+ is fully regular, it is also normal. Converse holds if X is weakly sequentially complete.*

We say that an ordered Banach space X is *lattice ordered* if $\sup\{x, y\}$ and $\inf\{x, y\}$ exist for all $x, y \in X$. Denote

$$|x| = \sup\{x, -x\}, \qquad x^+ = \sup\{0, x\} \quad \text{and} \quad x^- = \sup\{-x, 0\} = (-x)^+, \quad x \in X. \tag{2.2}$$

X is called a *Banach lattice* if

$$|x| \leqslant |y| \quad \text{in } X \quad \text{implies} \quad \|x\| \leqslant \|y\|. \tag{2.3}$$

Next we study Henstock–Kurzweil integrability of functions from a compact real interval $[a, b]$ to a Banach space X.

We say that $D = \{(\xi_i, I_i)\}$ is a *K-partition of* $[a, b]$ if $\{I_i\}$ is a finite collection of closed intervals I_i whose union is $[a, b]$ and which are non-overlapping, i.e. their interiors are pairwise disjoint, and if $\xi_i \in I_i$ for every i. D is called a *partial K-partition* of $[a, b]$ if $\bigcup_i I_i$ is a proper subset of $[a, b]$. Given a function δ from $[a, b]$ to $(0, \infty)$, called *gauge* of $[a, b]$, we say that a K-partition $D = \{(\xi_i, I_i)\}$ is *δ-fine* if $I_i \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for every i. The length of I_i is denoted by $|I_i|$.

A function $u : [a, b] \to X$ is *Henstock–Kurzweil* (shortly HK) *integrable* if there exists an element of X, denoted by ${}^K\!\int_a^b u(s)\,ds$ and called a *Henstock–Kurzweil integral* of u over $[a, b]$, having the following property: For every $\epsilon > 0$ there is such a gauge δ of $[a, b]$ that

$$\left\| \sum_i u(\xi_i)|I_i| - {}^K\!\!\int_a^b u(s)\,ds \right\| < \epsilon$$

whenever $D = \{(\xi_i, I_i)\}$ is a δ-fine K-partition of $[a, b]$.

If A is a subset of $[a, b]$, χ_A is the characteristic function of A, and if $\chi_A f$ is HK integrable on $[a, b]$, we say that f is HK integrable on A, and define

$${}^K\!\!\int_A f(s)\,ds = {}^K\!\!\int_a^b \chi_A(s) f(s)\,ds.$$

If f is HK integrable on $[a, b]$, it is HK integrable on every subinterval of $[a, b]$, and the Henstock–Kurzweil integral of f is linear and additive over non-overlapping subintervals of $[a, b]$ (cf. [15]).

The proofs for the results of the next lemma can be found, e.g., from [15].

Lemma 2.2.

(a) *The a.e. equal functions are HK integrable and their integrals are equal if one of these functions is HK integrable.*
(b) *A Bochner integrable function $f : [a, b] \to X$ is HK integrable, and $\int_I f(s)\,ds = {}^K\!\int_I f(s)\,ds$ whenever I is a closed subinterval of $[a, b]$.*

The next result plays an important role in the theory of HK integrable functions in ordered Banach spaces.

Lemma 2.3. *Let X be an ordered Banach space, and let $f_\pm : [a,b] \to X$ be HK integrable. If $f_-(s) \leqslant f_+(s)$ for a.e. $s \in [a,b]$, and if I is a closed subinterval of $[a,b]$, then*

$$ {}^K\!\!\int_I f_-(s)\,ds \leqslant {}^K\!\!\int_I f_+(s)\,ds. \tag{2.4}$$

Proof. By Lemma 2.2(a) we may assume that $f_-(s) \leqslant f_+(s)$ for all $s \in [a,b]$. Denoting $f = f_+ - f_-$, then $f(s)$ belongs to the order cone X_+ of X for all $s \in [a,b]$. Let $I = [c,d]$ be a closed subinterval of $[a,b]$. f is HK integrable on I. To prove that ${}^K\!\!\int_I f(s)\,ds \in X_+$, notice first that ${}^K\!\!\int_I f(s)\,ds = 0 \in X_+$ if $c = d$. Assume next that $c < d$. According to the definition of HK integrability we can choose for each $n \in \mathbb{N}$ a function $\delta_n : [c,d] \to (0,\infty)$, partitions $\{t_i^n\}_{i=1}^{m_n}$ of $[c,d]$ and points ξ_i^n so that $\xi_i^n \in [t_{i-1}^n, t_i^n] \subset (\xi_i^n - \delta(\xi_i^n), \xi_i^n + \delta(\xi_i^n))$, and that

$$\left\| \sum_{i=1}^{m_n} {}^K\!\!\int_I f(s)\,ds - f\big(\xi_i^n\big)\big(t_i^n - t_{i-1}^n\big) \right\| < \frac{1}{n}.$$

Denoting $y_n = \sum_{i=1}^{m_n} f(\xi_i^n)(t_i^n - t_{i-1}^n)$, $n \in \mathbb{N}$, we obtain

$$\left\| {}^K\!\!\int_I f(s)\,ds - y_n \right\| = \left\| {}^K\!\!\int_I f(s)\,ds - \sum_{i=1}^{m_n} f\big(\xi_i^n\big)\big(t_i^n - t_{i-1}^n\big) \right\| < \frac{1}{n}, \quad n \in \mathbb{N}.$$

Thus ${}^K\!\!\int_I f(s)\,ds = \lim_{n\to\infty} y_n \in X_+$, since X_+ is closed, and since $y_n \in X_+$ for every $n \in \mathbb{N}$. Consequently,

$$0 \leqslant {}^K\!\!\int_I f(s)\,ds = {}^K\!\!\int_I f_+(s)\,ds - {}^K\!\!\int_I^t f_-(s)\,ds.$$

This proves the assertion. □

3. Volterra integral equation

In this section we assume that X is a weakly sequentially complete Banach lattice. We shall study the Volterra integral equation

$$u(t) = h(t,u) + {}^K\!\!\int_a^t g\big(s, u(s), u\big)\,ds, \quad t \in J := [a,b], \tag{3.1}$$

where $h : J \times L^1(J,X) \to X$ and $g : J \times X \times L^1(J,X) \to X$. We shall prove existence and comparison results for Eq. (3.1) when h and g satisfy the following hypotheses:

(h) $h(t,u)$ is increasing in u for a.e. $t \in J$, strongly measurable in t for all $u \in L^1(J,X)$, and there exists an $\alpha \in L^1(J,\mathbb{R}_+)$ such that $\|h(t,u)\| \leqslant \alpha(t)$ for a.e. $t \in J$ and for all $u \in L^1(J,X)$.
(ga) $g(\cdot, u(\cdot), u)$ is HK integrable for all $u \in L^1(J,X)$.
(gb) If $u \leqslant v$ in $L^1(J,X)$, then $g(s,u(s),u) \leqslant g(s,v(s),v)$ for a.e. $s \in J$.
(gc) The sequence $({}^K\!\!\int_a^b g(s,u_n(s),u_n)\,ds)_{n=0}^\infty$ is bounded whenever $(u_n)_{n=0}^\infty$ is a monotone sequence in $L^1(J,X)$.

In the proof of our main result we use a special case of the following fixed point theorem.

Theorem 3.1. *Let P be a subset of an ordered normed space which has an order center c, i.e.,* $\sup\{c,u\}$ *and* $\inf\{c,u\}$ *exist and belong to P for every $u \in P$. Assume that $G : P \to P$ is increasing and maps monotone sequences to convergent sequences. Then*

(a) *G has minimal and maximal fixed points*;
(b) *G has the smallest and greatest fixed points u_* and u^* in the order interval $[\underline{u}, \overline{u}] = \{u \in P\colon \underline{u} \leqslant u \leqslant \overline{u}\}$, where $\underline{u}$ is the greatest solution of $u = \inf\{c, Gu\}$ and $\overline{u}$ is the smallest solution of $u = \sup\{c, Gu\}$.*

Moreover, all the solutions $\underline{u}$, $\overline{u}$, u_* and u^* are increasing with respect to G.

Proof. Although only ordinary sequences appear in the hypotheses, chain methods are needed to prove the asserted results. They follow from the results of [6, Proposition 2.1] and [7, Theorem 2.1], and from their proofs, provided that the following condition is valid.

(G) $\sup G[C]$ (respectively $\inf G[C]$) exists in P whenever C is a well-ordered (respectively an inversely well-ordered) chain in P.

To prove (G), let C be a well-ordered chain in P. Since G is increasing, then $G[C]$ is a well-ordered chain in P. Let $(v_n)_{n=0}^{\infty}$ be an increasing sequence in $G[C]$. Denoting $u_n = \min\{u \in C\colon\ Gu = v_n\}$, $n \in \mathbb{N}$, we obtain an increasing sequence $(u_n)_{n=0}^{\infty}$ of C, and $v_n = Gu_n$, $n \in \mathbb{N}$. Thus $(v_n)_{n=0}^{\infty}$ has a limit in P by a hypothesis. It then follows from [10, Proposition 1.1.5] that $\sup G[C]$ exists in P. The proof that $\inf G[C]$ exists in P whenever C is an inversely well-ordered chain in P is dual to the above proof. □

As a consequence of Theorem 3.1 which is also a special case of [2, Proposition 2.40], we obtain the following fixed point result.

Lemma 3.1. *Assume that $G : L^1(J, X) \to L^1(J, X)$ is increasing with respect to the a.e. pointwise ordering and maps monotone sequences to convergent sequences of $(L^1(J, X), \|\cdot\|_1)$. Then*

(a) *G has minimal and maximal fixed points*;
(b) *G has the smallest and greatest fixed points u_* and u^* in the order interval $[\underline{u}, \overline{u}]$ of $L^1(J, X)$, where $\underline{u}$ is the greatest solution of $u = -(-Gu)^+$ and $\overline{u}$ is the smallest solution of $u = (Gu)^+$.*

Moreover, all the solutions $\underline{u}$, $\overline{u}$, u_* and u^* are increasing with respect to G.

Proof. Since X is a Banach lattice, it can be shown (see, e.g., [14]) that $|x^{\pm} - y^{\pm}| \leqslant |x - y|$ for all $x, y \in X$. Thus, by (2.3),

$$\left\|x^{\pm} - y^{\pm}\right\| \leqslant \|x - y\|, \quad x, y \in X.$$

In particular, the mappings $X \ni x \mapsto x^{\pm}$ are continuous. Hence, if $u \in L^1(J, X)$, then the mappings $u^{\pm} := t \mapsto u(t)^{\pm}$ belong to $L^1(J, X)$. Consequently, the zero function is an order center of $L^1(J, X)$. Moreover, $L^1(J, X)$ is an ordered normed space. Noticing also that $\sup\{0, u\} = u^+$ and $\inf\{0, u\} = -(-u)^+$, the conclusions follow from Theorem 3.1. □

As an application of Lemma 3.1 we shall next prove existence and comparison results for the Volterra integral equation.

Theorem 3.2. *Assume that* (h), (ga), (gb) *and* (gc) *are valid. Eq.* (3.1) *has*

(a) *minimal and maximal solutions*;
(b) *the smallest and greatest solutions u_* and u^* in the order interval $[\underline{u}, \overline{u}]$ of $L^1(J, X)$, where $\underline{u}$ is the greatest solution of the equation*

$$u(t) = -\left(-h(t, u) - {}^{K}\!\!\int_a^t g\big(s, u(s), u\big)\, ds\right)^+, \quad t \in J,$$

and $\overline{u}$ is the smallest solution of the equation

$$u(t) = \left(h(t, u) + {}^{K}\!\!\int_a^t g\big(s, u(s), u\big)\, ds\right)^+, \quad t \in J.$$

Moreover, all the solutions $\underline{u}$, $\overline{u}$, u_* and u^* are increasing with respect to g and h.

Proof. According to the hypotheses (h), (ga) and (gb) the relation

$$Gu(t) = h(t, u) + {}^{K}\!\!\int_a^t g\big(s, u(s), u\big)\, ds, \quad t \in J, \tag{3.2}$$

defines mapping $G : L^1(J, X) \to L^1(J, X)$ which is increasing by Lemma 2.3. To prove that G maps monotone sequences to convergent sequences, let $(u_n)_{n=0}^{\infty}$ be an increasing sequence in $L^1(J, X)$. Denoting

$$v_n(t) = g\big(t, u_n(t), u_n\big), \qquad w_n(t) = {}^{K}\!\!\int_a^t v_n(s)\, ds, \quad t \in J,\ n \in \mathbb{N}, \tag{3.3}$$

we get increasing sequences of HK integrable functions $v_n : J \to X$, and continuous functions $w_n : J \to X$. Thus

$$0 \leqslant w_m(t) - w_n(t) = {}^K\!\!\int_a^t \big(v_m(s) - v_n(s)\big)\,ds \leqslant {}^K\!\!\int_a^b \big(v_m(s) - v_n(s)\big)\,ds$$

whenever $t \in J$, and $n \leqslant m$. This result and (2.1) imply that

$$\big\|w_m(t) - w_n(t)\big\| \leqslant \left\| {}^K\!\!\int_a^b \big(v_m(s) - v_n(s)\big)\,ds \right\|, \quad n \leqslant m. \tag{3.4}$$

The hypothesis (gc) implies that the sequence $({}^K\!\!\int_a^b v_n(s)\,ds)_{n=0}^\infty$ is bounded in X. It is also increasing by Lemma 2.3. Moreover, the order cone of X is normal by (2.1), and hence fully regular by Lemma 2.1. Thus the sequence $({}^K\!\!\int_a^b v_n(s)\,ds)_{n=0}^\infty$ converges, whence it is a Cauchy sequence in X. This result and (3.4) imply that $(w_n)_{n=0}^\infty$ converges uniformly on J, and hence also in $(L^1(J,X), \|\cdot\|_1)$.

In view of the hypothesis (h) the sequence $(h(\cdot,u_n))_{n=0}^\infty$ is increasing and a.e. pointwise bounded by $\alpha \in L^1(J,\mathbb{R}_+)$. Thus $(h(\cdot,u_n))_{n=0}^\infty$ converges a.e. pointwise since the order cone of X is fully regular. These results and the dominated convergence theorem ensure that $(h(\cdot,u_n))_{n=0}^\infty$ converges in $(L^1(J,X), \|\cdot\|_1)$.

Since $Gu_n(t) = h(t,u_n) + w_n(t)$ for all $t \in J$ and $n \in \mathbb{N}$, the above proof shows that the sequence $(Gu_n)_{n=0}^\infty$ converges in $(L^1(J,X), \|\cdot\|_1)$ whenever $(u_n)_{n=0}^\infty$ is an increasing sequence in $L^1(J,X)$.

The proof that $(Gu_n)_{n=0}^\infty$ converges in $(L^1(J,X), \|\cdot\|_1)$ whenever $(u_n)_{n=0}^\infty$ is a decreasing sequence in $L^1(J,X)$ is similar.

The above proof shows that G satisfies the hypotheses of Lemma 3.1. Since the solutions of (3.1) are same as the fixed points of G, the conclusions follow from Lemma 3.1. □

4. Cauchy problem

In this section the results of Section 3 are applied to Cauchy problems. Let X be a weakly sequentially complete Banach lattice. Consider first the functional impulsive Cauchy problem (ICP)

$$\begin{cases} u'(t) = g\big(t, u(t), u\big) \quad \text{a.e. on } J := [a,b], \\ u(a) = x_0, \quad \Delta u(\lambda) = D(\lambda, u), \quad \lambda \in W, \end{cases} \tag{4.1}$$

where $g : J \times X \times L^1(J,X) \to X$, $x_0 \in X$, $\Delta u(\lambda) = u(\lambda + 0) - u(\lambda)$, $D : W \times L^1(J,X) \to X$, and W is a well-ordered (and hence countable) subset of (a,b).

Definition 4.1. Denoting $W^{<t} = \{\lambda \in W \colon \lambda < t\}$, $t \in J$, we say that $u : J \to X$ is a mild solution of the ICP (4.1) if it belongs to the set

$$\begin{cases} V = \Big\{u : J \to X \colon \sum_{\lambda \in W} \big\|\Delta u(\lambda)\big\| < \infty \text{ and } t \mapsto u(t) - \sum_{\lambda \in W^{<t}} \Delta u(\lambda) \in P\Big\}, \\ \text{where} \\ P = \Big\{v : J \to X \colon v(t) - v(a) = {}^K\!\!\int_a^t w(s)\,ds,\ t \in J,\ w \in HK(J,X)\Big\}, \end{cases}$$

and if u satisfies the Volterra integral equation

$$u(t) = x_0 + \sum_{\lambda \in W^{<t}} D(\lambda, u) + {}^K\!\!\int_a^t g\big(s, u(s), u\big)\,ds, \quad t \in J. \tag{4.2}$$

It is easy to verify that V is a subset of $L^1(J,X)$.

The following result justifies the above definition.

Lemma 4.1. *If $q \in HK(J,X)$, $x_0 \in X$ and $c : W \to X$, and if $\sum_{\lambda \in W} \|c(\lambda)\| < \infty$, then the following results are valid.*

(a) *The equation*

$$u(t) = x_0 + \sum_{\lambda \in W^{<t}} c(\lambda) + {}^K\!\!\int_a^t q(s)\,ds, \quad t \in J, \tag{4.3}$$

defines a function $u \in V$.

(b) *u is a unique mild solution of the problem*

$$u'(t) = q(t) \quad \text{a.e. on } J, \qquad u(a) = x_0, \qquad \Delta u(\lambda) = c(\lambda), \quad \lambda \in W. \tag{4.4}$$

(c) *u is increasing with respect to q, c and x_0.*

(d) *For every $x^* \in X^*$ there is a null-set Z in J, which may depend on the choice of x^*, such that*

$$\big(x^*(u)\big)'(t) = x^*\big(q(t)\big) \quad \text{for all } t \in J \setminus Z. \tag{4.5}$$

Proof. (a) Let $u : J \to X$ be defined by (4.3). Define a mapping $\Gamma : J \to J$ by

$$\Gamma(s) = \min\big\{t \in W \cup \{b\}\colon s < t\big\}, \quad s \in [a, b), \qquad \Gamma(b) = b.$$

Denote by C the well-ordered chain of Γ-iterations of a, i.e. (cf. [10, Theorem 1.1.1]) C is the only well-ordered subset of J with the following properties:

$$a = \min C, \quad \text{and} \quad \text{if } s > a, \quad \text{then } s \in C \quad \text{iff} \quad s = \sup \Gamma\{t \in C \mid t < s\}.$$

It follows from [10, Corollary 1.1.1] that $W \subset C$, and J is a disjoint union of C and open intervals $(s, \Gamma(s))$, $s \in C$. Moreover, C is countable as a well-ordered set of real numbers, and c is constant on every interval $(s, \Gamma(s))$, $s \in C$. Hence

$$u(t) - u(\bar{t}) = \int_{\bar{t}}^t q(\tau)\,d\tau, \quad s < \bar{t} < t < \Gamma(s),\ s \in C. \tag{4.6}$$

For each $\lambda \in W$ the open interval $(\lambda, \Gamma(\lambda))$ does not contain any point of W, so that

$$\Delta u(\lambda) = u(\lambda + 0) - u(\lambda) = \lim_{t \to \lambda+0} c(\lambda) + {}^K\!\!\int_\lambda^t q(s)\,ds = c(\lambda), \quad \lambda \in W. \tag{4.7}$$

The above proof shows that u belongs to V.

(b) If $v \in V$ is a mild solution of (4.4), then $u - v$ is a function of V, $u(a) - v(a) = 0$, and $\Delta u(\lambda) - \Delta v(\lambda) = 0$ for each $\lambda \in W$. This result and the fact that u and v are mild solutions of (4.4) imply that $u(t) - v(t) = {}^K\!\!\int_a^t q(s)\,ds - {}^K\!\!\int_a^t q(s)\,ds = 0$ for every $t \in J$. This proves the uniqueness.

(c) Applying Lemma 2.3 and the representation (4.3) of u we see that u is increasing with respect to q, c and x_0.

(d) Let $x^* \in X^*$ be given. It follows from (4.6) by [15, Theorem 7.4.20] that

$$\big(x^*(u)\big)'(t) = x^*\big(q(t)\big) \quad \text{for a.e. } t \in \big(s, \Gamma(s)\big),\ s \in C. \tag{4.8}$$

This result implies the last assertion because J is a disjoint union of the countable subset C of J and open intervals $(s, \Gamma(s))$, $s \in C$. $\square$

As an application of Lemma 3.1 we prove an existence and comparison result for the smallest and greatest mild solutions of problem (4.1).

Given a well-ordered subset W of (a, b), assume that $D : W \times L^1(J, X) \to X$ satisfies the following hypotheses.

(D0) $D(\lambda, u)$ is increasing in u for all $\lambda \in W$, and there exists a $c : W \to X$ such that $\|D(\lambda, u)\| \leqslant c(\lambda)$ for all $\lambda \in W$ and $u \in L^1(J, X)$, and that $\sum_{\lambda \in W} \|c(\lambda)\| < \infty$.

The hypotheses given for D ensure that for each $x_0 \in X$ the relation

$$h(t, u) = x_0 + \sum_{\lambda \in W^{<t}} D(\lambda, u), \quad t \in J,\ u \in L^1(J, X), \tag{4.9}$$

defines a mapping $h : J \times L^1(J, X) \to X$ which satisfies the hypothesis (h) of Theorem 3.2. Then the integral equation (3.1) can be rewritten by (4.9) as a fixed point equation

$$u(t) = Gu(t) := x_0 + \sum_{\lambda \in W^{<t}} D(\lambda, u) + {}^K\!\!\int_a^t g\big(s, u(s), u\big)\, ds. \tag{4.10}$$

The next result is a consequence of Lemma 3.1.

Proposition 4.1. *Assume that the hypotheses* (ga), (gb), (gc) *and* (D0) *hold. Then the problem* (4.1) *has*

(a) *minimal and maximal mild solutions*;
(b) *the smallest and greatest mild solutions u_* and u^* in $[\underline{u}, \overline{u}] = \{u \in L^1(J, X)\colon\ \underline{u} \leqslant u \leqslant \overline{u}\}$, where $\underline{u}$ is the greatest solution of the equation*

$$u(t) = -\left(-x_0 - \sum_{\lambda \in W^{<t}} D(\lambda, u) - {}^K\!\!\int_a^t g\big(s, u(s), u\big)\, ds\right)^+, \quad t \in J,$$

and $\overline{u}$ is the smallest solution of the equation

$$u(t) = \left(x_0 + \sum_{\lambda \in W^{<t}} D(\lambda, u) + {}^K\!\!\int_a^t g\big(s, u(s), u\big)\, ds\right)^+, \quad t \in J.$$

Moreover, all the solutions $\underline{u}$, $\overline{u}$, u_* and u^* are increasing with respect to f, D and x_0.

Consider next the non-impulsive case, i.e. the Cauchy problem

$$u'(t) = g\big(t, u(t), u\big) \quad \text{a.e. on } J := [a, b], \qquad u(a) = x_0. \tag{4.11}$$

Any mild solution of (4.11) satisfies the integral equation

$$u(t) = x_0 + {}^K\!\!\int_a^t g\big(s, u(s), u\big)\, ds, \quad t \in J. \tag{4.12}$$

Every solution of (4.12) is continuous by [15, Theorem 7.4.1]. Thus we replace $L^1(J, X)$ by the space $C(J, X)$ of continuous functions from J to X.

We are going to show that the Cauchy problem (4.11) has for each $x_0 \in X$ mild solutions if $g : J \times X \times C(J, X) \to X$ satisfies the following hypotheses.

(g0) $g(\cdot, u(\cdot), u)$ is HK integrable for all $u \in C(J, X)$.
(g1) $g(t, x, u)$ is increasing with respect to x and u for a.e. $t \in J$.
(g2) If $(u_n)_{n=0}^\infty$ is a monotone sequence in $C(J, X)$, then $({}^K\!\!\int_a^b g(s, u_n(s), u_n)\, ds)_{n=0}^\infty$ is a bounded sequence of X.

The next result is a consequence of Lemma 3.1.

Proposition 4.2. *Assume that the hypotheses* (g0)–(g2) *hold. Then the CP* (4.11) *has*

(a) *minimal and maximal mild solutions*;
(b) *the smallest and greatest mild solutions u_* and u^* in the order interval $[\underline{u}, \overline{u}]$ of $C(J, X)$, where $\underline{u}$ is the greatest solution of the equation*

$$u(t) = -\left(-x_0 - {}^K\!\!\int_a^t g\big(s, u(s), u\big)\, ds\right)^+, \quad t \in J,$$

and $\overline{u}$ is the smallest solution of the equation

$$u(t) = \left(x_0 + {}^K\!\!\int_a^t g\big(s, u(s), u\big)\, ds\right)^+, \quad t \in J.$$

Moreover, all the solutions $\underline{u}$, $\overline{u}$, u_* and u^* are increasing with respect to f and x_0.

Proof. According to the hypotheses (g0) and (g1) the relation

$$Gu(t) = x_0 + {}^K\!\!\int_a^t g\big(s, u(s), u\big)\, ds, \quad t \in J, \tag{4.13}$$

defines mapping $G : C(J, X) \to C(J, X)$ which is increasing by Lemma 2.3. The proofs that G maps monotone sequences of $C(J, X)$ with respect to pointwise ordering to convergent sequences with respect to the uniform norm of $C(J, X)$, and that zero mapping is an order center of $C(J, X)$ are similar to those given in Theorem 3.2. Thus the hypotheses of Lemma 3.1 hold for G, defined by (3.2), when $L^1(J, X)$ is replaced by $C(J, X)$. Notice also that with this replacement the conclusions of Lemma 3.1 hold. Since the mild solutions of (4.11) are same as the fixed points of G, the conclusions follow from the conclusions of Lemma 3.1. □

Example 4.1. Denote

$$l_2(\mathbb{N} \times \mathbb{N}) = \left\{ x = (x_{i,j})_{i,j=0}^{\infty}\colon\ x_{i,j} \in \mathbb{R},\ \|x\|_2 = \left(\sum_{i,j=0}^{\infty} |x_{i,j}|^2 \right)^{\frac{1}{2}} < \infty \right\}. \tag{4.14}$$

The Banach space $X := (l_2(\mathbb{N} \times \mathbb{N}), \|\cdot\|_2)$ is weakly sequentially complete Banach lattice with respect to the ordering:

$$x = (x_{i,j})_{i,j=0}^{\infty} \leqslant y = (y_{i,j})_{i,j=0}^{\infty} \quad \text{iff} \quad x_{i,j} \leqslant y_{i,j} \quad \text{for all } (i, j) \in \mathbb{N} \times \mathbb{N}.$$

The characteristic functions $e_{i,j}$ of the singletons $\{(i, j)\}$ in $\mathbb{N} \times \mathbb{N}$ form an orthonormal basis for $l_2(\mathbb{N} \times \mathbb{N})$. Define functions $g_i : [0, 1] \to l_2(\mathbb{N} \times \mathbb{N})$, $i \in \mathbb{N}$, by

$$g_0(t) = \begin{cases} (0)_{i,j=0}^{\infty}, & t = 0, \\ \left(\frac{1}{(i+1)(j+1)}\left(2t\cos\left(\frac{1}{t^2}\right) + \frac{2}{t}\sin\left(\frac{1}{t^2}\right)\right)\right)_{i,j=0}^{\infty}, & t \in (0, 1], \end{cases} \tag{4.15}$$

and

$$g_i(t) = \begin{cases} 2^i e_{i,j}, & \frac{j}{2^i} \leqslant t < \frac{j}{2^i} + \frac{1}{2^{2i}},\ i = 1, 2, \ldots,\ j = 0, \ldots, 2^i - 1, \\ 0 & \text{otherwise}. \end{cases} \tag{4.16}$$

The functions $f_n : [0, 1] \to l_2(\mathbb{N} \times \mathbb{N})$, $n \in \mathbb{N}$, defined by

$$f_n(t) = \sum_{i=0}^{n} g_i(t), \quad t \in [0, 1],\ n \in \mathbb{N}, \tag{4.17}$$

form an increasing sequence of HK integrable functions. There exists by [8, Example 3.1] such an HK integrable function $f : [0, 1] \to X$ that $f(s) = \lim_n f_n(s)$ for a.e. $s \in [0, 1]$, and that $\lim_{n\to\infty} {}^K\!\!\int_0^1 f_n(s)\, ds = {}^K\!\!\int_0^1 f(s)\, ds$.

Let $f : [0, 1] \to l_2(\mathbb{N} \times \mathbb{N})$ be the HK integrable function constructed above, let $h_1 : l_2(\mathbb{N} \times \mathbb{N}) \to l_2(\mathbb{N} \times \mathbb{N})$ be increasing, continuous and bounded, and let $h_2 : C([0, 1], l_2(\mathbb{N} \times \mathbb{N})) \to l_2(\mathbb{N} \times \mathbb{N})$ be increasing and bounded. Then the equation

$$g(t, x, u) = f(t) + h_1(x) + h_2(u), \quad t \in J := [0, 1],\ x \in l_2(\mathbb{N} \times \mathbb{N}),\ u \in C\big(J, l_2(\mathbb{N} \times \mathbb{N})\big), \tag{4.18}$$

defines a function $g : J \times l_2(\mathbb{N} \times \mathbb{N}) \times C(J, l_2(\mathbb{N} \times \mathbb{N})) \to l_2(\mathbb{N} \times \mathbb{N})$ which satisfies the hypotheses (g0)–(g2) when $X = l_2(\mathbb{N} \times \mathbb{N})$. Thus the Cauchy problem

$$u'(t) = f(t) + h_1\big(u(t)\big) + h_2(u), \qquad u(0) = x_0 \tag{4.19}$$

has by Proposition 4.2 for each $x_0 \in l_2(\mathbb{N} \times \mathbb{N})$

(a) minimal and maximal mild solutions;
(b) the smallest and greatest mild solutions u_* and u^* in $[\underline{u}, \overline{u}] = \{u \in C(J, l_2(\mathbb{N} \times \mathbb{N}))\colon\ \underline{u} \leqslant u \leqslant \overline{u}\}$, where $\underline{u}$ is the greatest solution of the equation

$$u(t) = -\left(-x_0 - {}^K\!\!\int_0^t \big(f(s) + h_1\big(u(s)\big) + h_2(u)\big)\, ds \right)^+, \quad t \in J,$$

and $\overline{u}$ is the smallest solution of the equation

$$u(t) = \left(x_0 + {}^K\!\!\int_0^t \big(f(s) + h_1\big(u(s)\big) + h_2(u)\big)\, ds \right)^+, \quad t \in J.$$

Moreover, all the solutions $\underline{u}$, $\overline{u}$, u_* and u^* are increasing with respect to f, g_1, g_2 and x_0.

Remark 4.1. By definition, a function u from a compact interval $[a,b]$ to a Banach space X is HK integrable if for every $\epsilon > 0$ there is such a gauge δ of $[a,b]$ that

$$\left\| \sum_i \left(u(\xi_i)|I_i| - {}^K\!\!\int_{I_i} u(s)\,ds \right) \right\| < \epsilon$$

whenever $D = \{(\xi_i, I_i)\}$ is a δ-fine K-partition of $[a,b]$. u is said to be HL integrable if for every $\epsilon > 0$ there is such a gauge δ of $[a,b]$ that

$$\sum_i \left\| u(\xi_i)|I_i| - {}^K\!\!\int_{I_i} u(s)\,ds \right\| < \epsilon$$

whenever $D = \{(\xi_i, I_i)\}$ is a δ-fine K-partition of $[a,b]$. According to [15, Theorem 3.6.5 and Proposition 3.6.6] every HL integrable function is HK integrable, and converse holds if X is finite-dimensional.

The function $g(\cdot, u(\cdot), u)$ is neither Bochner nor HL integrable for any choice of $u \in C(J, l_2(\mathbb{N} \times \mathbb{N}))$ when g is defined by (4.18), because f is not HL integrable (see [3, Example 3.2]).

It follows from [11, Corollary 4.1] that if $u, v : J \to X$ are HK integrable, and $u \leqslant v$, and if one of them is McShane integrable, then both are McShane integrable. In particular, all the results of Sections 3, 4 and 5 remain valid if HK integrability is replaced by McShane integrability.

The following spaces are examples of weakly sequentially complete Banach lattices:

– A reflexive Banach lattice.
– A Banach lattice which is uniformly monotone in the sense defined in [1, XV, 14].
– $\mathbb{R}^m$, ordered coordinatewise and normed by a p-norm, $1 \leqslant p < \infty$.
– A separable Hilbert space whose order cone is generated by an orthonormal basis.
– A sequence space l^p, $1 \leqslant p < \infty$, normed by p-norm and ordered componentwise.
– A function space $L^p(\Omega)$, $1 \leqslant p < \infty$, normed by p-norm and ordered a.e. pointwise, where Ω is a measure space.
– A function space $L^p([a,b], X)$, $1 \leqslant p < \infty$, ordered a.e. pointwise, where X is any of the spaces listed above.

In [8] existence results are derived for the smallest and greatest solutions of a discontinuous functional Urysohn integral equation in ordered Banach spaces containing HK integrable functions. Discontinuous functional differential and integral equations in ordered Banach spaces containing HL integrable or Bochner integrable functions are studied, e.g., in [2, Sections 6 and 7] (see also references therein). As for other results on integral equations including non-absolutely integrable Banach space-valued functions, see, e.g., [4,9,13,16,17]. Compared with these papers a novelty of the results of Sections 3 and 4 is that no continuity hypotheses are imposed on the function g. Moreover, no hypothesis on the existence of sub- and/or supersolutions of considered equations is needed.

References

[1] G. Birkhoff, Lattice Theory, Amer. Math. Soc. Colloq. Publ., vol. XXV, Amer. Math. Soc., Rhode Island, 1940.
[2] S. Carl, S. Heikkilä, Fixed Point Theory in Ordered Spaces and Applications, Springer, Berlin, Heidelberg, New York, ISBN 978-1-4419-7584-3, 2011.
[3] M. Federson, Some peculiarities of the Henstock and Kurzweil integrals of Banach space-valued functions, Real Anal. Exchange 29 (1) (2003/2004) 1–22.
[4] M. Federson, M. Bianconi, Linear Fredholm integral equations and the integral of Kurzweil, J. Appl. Anal. 8 (1) (2002) 81–108.
[5] D. Guo, Y.J. Cho, J. Zhu, Partial Ordering Methods in Nonlinear Problems, Nova Science Publishers, New York, 2004.
[6] S. Heikkilä, Existence and comparison results for operator and differential equations in abstract spaces, J. Math. Anal. Appl. 274 (2002) 586–607.
[7] S. Heikkilä, Existence results for operator equations in abstract spaces and an application, J. Math. Anal. Appl. 292 (2004) 262–273.
[8] S. Heikkilä, Monotone convergence theorems for Henstock–Kurzweil integrable functions and applications, J. Math. Anal. Appl. 377 (1) (2011) 286–295.
[9] S. Heikkilä, M. Kumpulainen, S. Seikkala, Convergence theorems for HL integrable vector-valued functions with applications, Nonlinear Anal. 70 (5) (2009) 1939–1955.
[10] S. Heikkilä, V. Lakshmikantham, Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations, Marcel Dekker, New York, 1994.
[11] S. Heikkilä, Guoju Ye, Convergence and comparison results for Henstock–Kurzweil and McShane integrable vector-valued functions, Southeast Asian Bull. Math. 34 (2010) 1–12.
[12] P.-Y. Lee, J. Lu, On singularity of Henstock integrable functions, Real Anal. Exchange 25 (2) (1999) 795–798.
[13] B.-R. Satco, Nonlinear Volterra integral equations in Henstock integrability setting, Electron. J. Differential Equations 39 (2008) 1–9.
[14] H.H. Schaefer, Topological Vector Spaces, Macmillan, New York, 1966.
[15] S. Schwabik, Guoju Ye, Topics in Banach Space Integration, World Scientific, Singapore, 2005.
[16] A. Sikorska-Nowak, On the existence of nonlinear integral equations in Banach spaces and Henstock–Kurzweil integrals, Ann. Polon. Math. 83 (3) (2004) 257–267.
[17] A. Sikorska-Nowak, Existence of solutions of nonlinear integral equations in Banach spaces and Henstock–Kurzweil integrals, Comment. Math. (Prace Mat.) 47 (2) (2007) 227–238.